TALK ON DRINFELD-GAITSGORY INTERPOLATION GRASSMANNIAN AND GEOMETRIC SATAKE EQUIVALENCE

1. Affine Grassmannian and geometric Satake

Let G be a reductive algebraic group over $\mathbb{C}$ and $T \subset G$ is a maximal torus. We assume for the latter that $[G, G]$ is simply connected (for simplicity). Set $\mathcal{K} := \mathbb{C}((z))$, $\mathcal{O} := \mathbb{C}[[z]]$. Define $\mathrm{Gr}_G := G(\mathcal{K})/G(\mathcal{O})$. This is and ind-projective scheme called affine Grassmannian of G. Let X be a smooth projective curve over $\mathbb{C}$ (for example $X = \mathbb{P}^1$). We fix a point $x \in X$ ($0 \in \mathbb{P}^1$). It is known that Gr_G has the following moduli description:

$$\mathrm{Gr}_G \simeq \{(\mathcal{E}, \sigma) \,|\, \sigma \colon \mathcal{E}|_{X \setminus \{x\}} \simeq \mathcal{E}^{\mathrm{triv}}|_{X \setminus \{x\}}\},$$

where $\mathcal{E}^{\mathrm{triv}}$ is a tr Consider the action $G(\mathcal{O}) \curvearrowright \mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$ via left multiplication (changing the trivialization).

Remark 1.1. Fix a maximal torus $T \subset G$, let Λ_G (Λ_G^+) be the lattice (submonoid) of cocharacters (dominant cocharacters) of $T \subset G$. Any $\lambda \in \Lambda_G$ gives rise to the element $z^\lambda \in \mathrm{Gr}_G$. We set $\mathrm{Gr}_G^\lambda := G(\mathcal{O}) \cdot z^\lambda$. We have the following (Cartan) decomposition:

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda_G^+} \mathrm{Gr}_G^\lambda .$$

Let $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ be the category of $G(\mathcal{O})$-equivariant perverse sheaves on Gr_G. This abelian category has a Tannakian structure – tensor product comes from the convolution and fiber functor simply sends a sheaf $\mathcal{P}$ to the global cohomology $V := H^*(\mathrm{Gr}_G, \mathcal{P})$. Geometric Satake isomorphism (to be denoted S_G) identifies $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ and $\mathrm{Rep}(G^\vee)$ as Tannakian categories, here $G^\vee$ is the Langlands dual group to G:

$$\begin{array}{ccc} \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) & \xrightarrow{\quad S_G \quad} & \mathrm{Rep}(G^\vee). \\ & {\scriptstyle H^*} \searrow \quad \swarrow {\scriptstyle \mathrm{Forg}} & \\ & \mathrm{Vect}_{\mathbb{C}} & \end{array}$$

It follows that for any $\mathcal{P} \in \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ there exists an action $\mathfrak{g}^\vee \curvearrowright H^*(\mathrm{Gr}_G, \mathcal{P})$. Goal: to construct it!

Remark 1.2. Let us recall the definition of a Tannakian category over $\mathbb{C}$. Let $(\mathcal{C}, \otimes)$ be an abelian tensor category over $\mathbb{C}$. We say that $\mathcal{C}$ is rigid if for any $V \in \mathcal{C}$, there exists a dual object V^* in the following sense: $\mathrm{Hom}_{\mathcal{C}}(W, V^*) \simeq \mathrm{Hom}_{\mathcal{C}}(W \otimes V, \mathbf{1})$ functorially on W, here $\mathbf{1}$ is the identity object of $(\mathcal{C}, \otimes)$. Let $F \colon \mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}$ be a tensor functor. We say that F is a fiber functor if it is exact and faithfull. We say that a category $\mathcal{C}$ is Tannakian if it is abelian, tensor, rigid, equipped with a fiber functor and such that $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = \mathbb{C}$.

Remark 1.3. It is enough to construct the action of $U(\mathfrak{n}^\vee)$ on $H^*(\mathrm{Gr}_G, \mathcal{P}) = V$. To see this, let us recall that the functor H^* is $T^\vee$-graded:

$$H^*(\mathrm{Gr}_G, -) \simeq \bigoplus_{\mu \in \Lambda_G} H_c^{\langle 2\rho^\vee, \mu\rangle}(S_\mu, -),$$

here $2\rho^\vee$ is the sum of positive roots. We obtain the action of $\mathfrak{t}^\vee \subset \mathfrak{g}^\vee$ on V. Let us now suppose that we have constructed the action $U(\mathfrak{n}^\vee) \curvearrowright V$ geometrically. It follows from the construction that if we interchange B and B_- then we will obtain the action of $U(\mathfrak{n}_-)$. More detailed: we have two decompositions (by $T^\vee$-weight spaces)

$$\bigoplus_\mu H_c^{\langle 2\rho^\vee, \mu\rangle)}(S_\mu, \mathcal{P}) \simeq V \simeq \bigoplus_\mu H_c^{\langle -2\rho^\vee, \mu\rangle}(T_\mu, \mathcal{P}).$$

The action of $U(\mathfrak{n}^\vee)$ comes from the first decomposition and the consideration of deformations of S_μ ($\overline{S}_\mu$), the action of $U(\mathfrak{n}_-^\vee)$ comes from the second decomposition and the consideration of deformations of T_μ ($\overline{T}_\mu$).

Remark 1.4. The convolution product of two perverse sheaves $\mathcal{P}_1$, $\mathcal{P}_2$ can be defined as follows. Consider the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{\mathbb{A}^1\times\mathbb{A}^1, G}$, and a diagonal embedding $\mathbb{A}^1 \hookrightarrow \mathbb{A}^1 \times \mathbb{A}^1$. Let U be the complement to the diagonal in $\mathbb{A}^1 \times \mathbb{A}^1$. It is easy to see that $\mathrm{Gr}_{\mathbb{A}^2,G}|_U \simeq (\mathrm{Gr}_{\mathbb{A}^1,G} \underset{\mathbb{A}^1}{\times} \mathrm{Gr}_{\mathbb{A}^1,G})|_U$. We consider the sheaf $\mathcal{P}_{\mathbb{A}^1,1} \underset{\mathbb{A}^1}{\boxtimes} \mathcal{P}_{\mathbb{A}^1,2}$ on $\mathrm{Gr}_{\mathbb{A}^2,G}|_U$. By the definition $\mathcal{P}_{\mathbb{A}^1,1} \underset{\mathbb{A}^1}{*} \mathcal{P}_{\mathbb{A}^1,2} := i^* j_{*!}(\mathcal{P}_{\mathbb{A}^1,1} \underset{\mathbb{A}^1}{\boxtimes} \mathcal{P}_{\mathbb{A}^1,2})$, where $j\colon \mathrm{Gr}_{\mathbb{A}^2,G}|_U \hookrightarrow \mathrm{Gr}_{\mathbb{A}^2,G}, i\colon \mathrm{Gr}_{\mathbb{A}^1,G} \hookrightarrow \mathrm{Gr}_{\mathbb{A}^2,G}$ are open and closed embeddings respectively. We set $\mathcal{P}_1 * \mathcal{P}_2 := (\mathcal{P}_{X,1} \underset{X}{*} \mathcal{P}_{X,2})|_{\mathrm{Gr}_G}$.

2. Geometric construction of the universal enveloping algebra

2.1. **Main problem.** Let B be a Borel subgroup of G, that contains T. Let Λ_G be the co-character lattice of $T \subset G$. Let B_- be the opposite Borel subgroup. Any $\mu \in \Lambda_G$ gives rise to the element $z^\mu \in \mathrm{Gr}_G$. For $\mu \in \Lambda_G$ define $S_\mu := U(\mathcal{K}) \cdot z^\mu$ (resp. $T_\mu := U_- \cdot z^\mu$).

Proposition 2.1. *(Braverman-Finkelberg-Gaitsgory-Mircovic)*
There exists an isomorphism $U(\mathfrak{n}^\vee) \simeq \bigoplus_{\mu\in\Lambda_G} H_c^{\mathrm{top}}(S_\mu \cap T_0, \mathbb{C}) =: \mathcal{A}$ *of $T^\vee$-graded vector spaces i.e. the vector space* $H_c^{\mathrm{top}}(S_\mu \cap T_0, \mathbb{C})$ *identifies with the $T^\vee$-weight space* $U(\mathfrak{n}^\vee)_\mu$.

It is known that $U(\mathfrak{n}^\vee)$ has a bialgebra structure. The goal for now is to describe multiplication and comultiplication morphisms in purely geometrical terms.

2.2. **Zastava spaces.** Suppose now that our projective curve X is $\mathbb{P}^1$. We fix a co-character $\mu \in \Lambda_G$. Set $\mathfrak{B} := G/B$ – the flag variety. Recall that $\mathrm{H}_2(\mathfrak{B}, \mathbb{C}) = \Lambda_G$, hence, we have a notion of a map $\mathbb{P}^1 \to \mathfrak{B}$ of degree μ. The equivalent way to define a degree is the following: recall that for any character $\lambda\colon T \to \mathbb{C}^\times$ one can consider the corresponding line bundle $\mathbf{L}_\lambda$ on $\mathfrak{B}$. We say that the map f is of degree μ if for any $\lambda \in \Lambda_G$ the pull back $f^*(\mathbf{L}_\lambda)$ has degree $\langle \lambda, \mu \rangle$. We denote

$$\overset{\circ}{\tilde{Z}}{}^\mu := \{f\colon \mathbb{P}^1 \to \mathfrak{B} \mid \deg(f) = \mu, f(\infty) = B_-\}.$$

It is known to be a smooth algebraic variety.

Example 2.2. Suppose $G = SL_2$, then $\mathfrak{B} \simeq \mathbb{P}^1$, B_- is ∞ and Λ_G identifies with $\mathbb{Z}$. We fix $n \in \mathbb{Z}$. The map from $\mathbb{P}^1$ to $\mathbb{P}^1$ of degree n that sends ∞ to ∞ is given by $(x : y) \mapsto (P(x, y) : Q(x, y))$ for some homogeneous polynomials P, Q of degree n having no common roots. We set $t := \frac{y}{x}$ and consider the corresponding polynomials $P(t), Q(t)$. The condition $f(\infty) = \infty$ is equivalent to $\deg(Q(t)) > \deg(P(t))$.

Remark 2.3. Set $k := \deg(Q(x, y)) = \deg(P(x, y))$. Morphism $(x : y) \mapsto (P(x, y) : Q(x, y)$ corresponds to the morphism $\mathcal{O}(-k) \to \mathcal{O} \oplus \mathcal{O}$ given by (P, Q), hence $k = n$.

It follows that the space $\overset{\circ}{\tilde{Z}}{}^n$ is isomorphic to the space

$$\{(P, Q) \mid \deg(Q) = n, \deg(P) < n, Q \text{ is monomial, } \gcd(P, Q) = 1\}.$$

If we drop the last condition we will obtain so-called Zastava space to be denoted $\tilde{Z}^n$ that is isomorphic to $\mathbb{A}^{2n}$ in our situation.

Let us point out that the map $\mathbb{P}^1 \to \mathbb{P}^1$ of degree n is the same as a line subbundle $\eta\colon \mathcal{L} \hookrightarrow \mathcal{O} \oplus \mathcal{O}$, hence, $\mathcal{L} \simeq \mathcal{O}(-n)$ and the morphism η is given by (P, Q). The condition $\gcd(P, Q) = 1$ precisely means that the morphism η is an embedding of line bundles.

Dropping this condition we obtain the following moduli space:

$$\{\eta\colon \mathcal{L} \hookrightarrow \mathcal{O} \oplus \mathcal{O} \mid \deg(\mathcal{L}) = -n, \eta \text{ is an embedding of coherent sheaves}\},$$

such that on the fiber at ∞ image of $\eta|_\infty$ is the line $(0, c) \subset \mathbb{C} \oplus \mathbb{C}$.

Let us also note that the space $\tilde{Z}^n$ comes equipped with the natural (factorization) morphism $\pi_n\colon \tilde{Z}^n \to S^n(\mathbb{A}^1) \simeq \mathbb{A}^1$ to the n-th symmetric power of $\mathbb{A}^1$. It sends (P, Q) to the set of roots of Q computed with multiplicities. More geometrically this is the divisor $f^{-1}(0) \subset \mathbb{P}^1$ (recall that f is our map from $\mathbb{P}^1$ to $\mathbb{P}^1$).

Let us now define zastava spaces for a general curve X (again we start from $G = SL_2$ case).

$$Z^n := \{\mathcal{L} \xrightarrow{\eta} \mathcal{E} \xrightarrow{\zeta} \mathcal{O} \mid \deg(\mathcal{L}) = -n,\ \zeta \circ \eta \text{ is generically isomorphism}, \Lambda^2(\mathcal{E}) \text{ is trivialized}.\}$$

When we say that the morphism $\zeta \circ \eta$ is generically isomorphism we mean that there are only finitely number of points of X in which our morphism is not an isomorphism (and hence zero). In other words it means that the morphism $\zeta \circ \eta\colon \mathcal{L} \to \mathcal{O}$ corresponds to a divisor $D \subset X$ of degree n. We obtain the factorization morphism $Z^n \to X^{(n)}$, where $X^{(n)} =: S^n(X)$.

Remark 2.4. The difference between Z^n and $\tilde{Z}^n$ is the following. For $X = \mathbb{P}^1$ we have the factorization morphism $Z^n \to (\mathbb{P}^1)^{(n)}$, after restricting it to $(\mathbb{A}^1)^{(n)}$ we obtain $\tilde{Z}^n$.

We can now proceed to the definition of Z^μ for arbitrary reductive group G and a cocharacter $\mu \in \Lambda_G$. We will do it via Tannakian formalism (in the spirit of the definition of the flag variety via functor of points and Plücker relations):

For $\lambda^\vee \in \Lambda^\vee$, let $\mathbb{C}^{\lambda^\vee}$ be the one-dimensional representation of T via character $\lambda^\vee\colon T \to \mathbb{C}^\times$. Let Z^μ be the moduli space of the following data:

1) a G-bundle $\mathcal{E}$ on X,

2) a T-bundle $\mathcal{T}$ on X of degree $-\mu$,

3) for every $\lambda^\vee \in \Lambda^{\vee+}$, a morphism of coherent sheaves $\mathbb{C}_{\mathcal{T}}^{\lambda^\vee} \xrightarrow{\eta_{\lambda^\vee}} \mathcal{V}_{\mathcal{E}}^{\lambda^\vee}$ and a surjective morphism of vector bundles $\mathcal{V}_{\mathcal{E}}^{\lambda^\vee} \xrightarrow{\zeta_{\lambda^\vee}} \mathcal{O}_{S\times X}$ satisfying the following conditions:

a) for every $\lambda^\vee \in \Lambda^{\vee+}$ the composition $(\zeta_{\lambda^\vee} \circ \eta_{\lambda^\vee})$ is an isomorphism generically,

the Plücker relations hold (this is exactly a reformulation of Plücker equations for flag varieties in families:

b) For every $\lambda^\vee, \mu^\vee \in \Lambda^{\vee+}$ let $\mathrm{pr}_{\lambda^\vee,\mu^\vee}\colon V^{\lambda^\vee} \otimes V^{\mu^\vee} \twoheadrightarrow V^{\lambda^\vee+\mu^\vee}$ be the projection morphism. We have the corresponding morphisms

$$\mathrm{pr}^{\mathcal{E}}_{\lambda^\vee,\mu^\vee}\colon \mathcal{V}_{\mathcal{E}}^{\lambda^\vee} \otimes \mathcal{V}_{\mathcal{E}}^{\mu^\vee} \to \mathcal{V}_{\mathcal{E}}^{\lambda^\vee+\mu^\vee}.$$

Then the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{O}_U \otimes \mathcal{O}_U & \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}} & \mathcal{O}_U \\ \downarrow{\scriptstyle \eta_{\lambda^\vee}\otimes\eta_{\mu^\vee}} & & \downarrow{\scriptstyle \eta_{\lambda^\vee+\mu^\vee}} \\ (\mathcal{V}_{\mathcal{E}}^{\lambda^\vee} \otimes \mathcal{V}_{\mathcal{E}}^{\mu^\vee})|_U & \xrightarrow{\mathrm{pr}^{\mathcal{E}}_{\lambda^\vee,\mu^\vee}|_U} & (\mathcal{V}_{\mathcal{E}}^{\lambda^\vee+\mu^\vee})|_U, \end{array}$$

$$\begin{array}{ccc} (\mathcal{V}_{\mathcal{E}}^{\lambda^\vee} \otimes \mathcal{V}_{\mathcal{E}}^{\mu^\vee})|_U & \xrightarrow{\mathrm{pr}^{\mathcal{E}}_{\lambda^\vee,\mu^\vee}|_U} & (\mathcal{V}_{\mathcal{E}}^{\lambda^\vee+\mu^\vee})|_U \\ \downarrow{\scriptstyle \zeta_{\lambda^\vee}\otimes\zeta_{\mu^\vee}} & & \downarrow{\scriptstyle \zeta_{\lambda^\vee+\mu^\vee}} \\ \mathcal{O}_U \otimes \mathcal{O}_U & \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}} & \mathcal{O}_U. \end{array}$$

c) Given a morphism $\mathrm{pr}\colon V^{\lambda^\vee} \otimes V^{\mu^\vee} \to V^{\nu^\vee}$ for $\lambda^\vee, \mu^\vee, \nu^\vee \in \Lambda^{\vee+},\ \nu^\vee < \lambda^\vee + \mu^\vee$, we have

$$\mathrm{pr}^{\mathcal{E}} \circ (\eta_{\lambda^\vee} \otimes \eta_{\mu^\vee}) = 0,\ (\zeta_{\lambda^\vee} \otimes \zeta_{\mu^\vee}) \circ \mathrm{pr}^{\mathcal{E}} = 0.$$

d) For $\lambda^\vee = 0$ we have $\zeta_{\lambda^\vee} = \mathrm{Id}$ and $\eta_{\lambda^\vee} = \mathrm{Id}$.

Remark 2.5. If we require morphisms $\eta_{\lambda^\vee}$ to be embeddings of vector bundles we will obtain the open zastava $\overset{\circ}{Z}{}^\mu$, if we do not require morphisms $\zeta_{\lambda^\vee}$ to be surjective we will obtain compactified zastava spaces $\overline{Z}^\mu$.

Remark 2.6. Let us give another definition of the spaces $\overset{\circ}{Z}{}^\mu$ (Z^μ). It is the moduli space of B-structures Ψ of degree μ in the trivial G-bundle $\mathcal{E}^{\mathrm{triv}}$ where $\mathcal{E}$ is a G-bundle, Ψ^+ is a B-structure of degree μ in $\mathcal{E}$, Ψ^- is a B_--structure of degree 0 in $\mathcal{E}$ such that these two structures are generically transversal.

Example 2.3. Again suppose $G = SL_2$. Then B-structure in the trivial SL_2-bundle is the same as an embedding $\eta\colon \mathcal{L} \hookrightarrow \mathcal{O} \oplus \mathcal{O}$ such that $\deg(\mathcal{L}) = n$. Standard U_--structure corresponds to the surjection $\zeta\colon \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}$ onto the second component. The transversality condition is simply a fact that $\pi \circ v$ is an isomorphism everywhere except the finite number of points. It follows that this morphism is isomorphic to

$\mathcal{O}(-D) \hookrightarrow \mathcal{O}$ for some divisor D. Thus, we obtain a morphism $\overset{\circ}{Z}{}^n \to S^n(\mathbb{A}^1)$. It is called a factorization morphism.

2.6.1. *Factorization morphism.* For $\alpha \in \Lambda_G^+$, let $\alpha = \sum n_i\alpha_i$. For a curve X, set $X^{(\alpha)} := \prod X^{(n_i)}$. In the same way as in the Example 2.2 we obtain a factorization morphism $\pi_\alpha \colon Z^\alpha \to X^{(\alpha)}$. Fix an element $D \in X^{(\alpha)}$, it can be written as $\sum \alpha_i \cdot D_i$, where $\deg(D_i) = n_i$. It follows that for any character $\lambda^\vee \in \Lambda_G^+$ we have a divisor $\sum \langle \lambda^\vee, \alpha_i\rangle D_i$. We will say that two elements $D, D' \in X^{(\alpha)}$ are disjoint if for any $\lambda^\vee$, the corresponding divisors are disjoint. Factorization morphism has the following property: for a decomposition $\alpha = \alpha_1 + \alpha_2$ and two disjoint divisors $D_1 \in X^{(\alpha_1)}, D_2 \in X^{(\alpha_2)}$ we have the natural isomorphism

$$\pi_{\alpha_1}^{-1}(D_1) \times \pi_{\alpha_2}^{-1}(D_2) \simeq \pi_\alpha^{-1}(D_1 + D_2).$$

Example 2.4. Again for $G = SL_2$ the factorization property simply says that any fraction $\frac{P}{Q_1 \cdot Q_2}$ with Q_1 and Q_2 having no common root can be uniquely presented as $\frac{P_1}{Q_1} + \frac{P_2}{Q_2}$.

Proposition 2.5. *Fix a point $x \in X$ and consider a divisor $\alpha \cdot x \in X^{(\alpha)}$, then the fiber $\mathcal{F}^\alpha := \pi_\alpha^{-1}(\alpha \cdot x)$ is isomorphic to the intersection $\overline{S}_\alpha \cap T_0$.*

Remark 2.7. Taking a fiber over $\alpha \cdot x$ for open zastava we obtain the intersection $S_\alpha \cap T_0$, for compactified zastava we get $\overline{S}_\alpha \cap \overline{T}_0$.

2.8. **Comultiplication via factorization.** Recall the morphism $\pi_\mu \colon Z^\mu \to X^{(\mu)}$. We fix a decomposition $\alpha = \alpha_1 + \alpha_2$ and consider the natural embedding

$$X \hookrightarrow X^{(\mu)},\ y \mapsto (\mu_1 \cdot x + \mu_2 \cdot y).$$

After restricting the factorization morphism to $\mathbb{A}^1$, we obtain a family $\pi_\mu^{-1}(X) \to X$ with zero fiber equal to $\mathcal{F}^\mu$ and the fiber over $y \in X \setminus \{x\}$ equal to $\mathcal{F}^{\mu_1} \times \mathcal{F}^{\mu_2}$.

2.9. **Cospecialization construction.** Let $\pi \colon S \to X$ be a one-parameter family over X, and assume that this family is trivial over $X \setminus \{0\}$ i.e. $\pi^{-1}(X \setminus \{x\}) \simeq S|_y \times (X \setminus \{x\})$ for some $y \neq x$. Then for any constructible complex F on S there exists a *cospecialization map* $H_c^*(S|_x, F|_x^*) \to H_c^*(S|_y, F|_y^*)$.

Remark 2.10. Cospecialization morphism is constructed as follows. Consider complex $\pi_!(F)$ on X. The restriction $\pi_!(F)|_{X \setminus \{x\}}$ identifies with $(\pi_! F|_1) \boxtimes \mathbb{C}_{X \setminus \{0\}}$. To construct the desired morphism we may suppose that F is a sheaf. It is also enough to construct a map $\pi_!(F)_0 \to \pi_!(F)_1$. We fix now an element $s \in \pi_!(F)_0$, it is represented by a section $s_U \in \pi_!(F)(U)$ for some open set containing $\{x\}$. Now we fix any point $t \in U$ such that $t \neq x$. The restriction $(s_U)_t$ defines an element of $\pi_!(F)|_t \simeq \pi_!(F)|_y$, hence, we obtain an element of $\pi_!(F)|_y$. Correctness is obvious.

Remark 2.11. More generally, given a stratified variety S and a complex F on S which is constant on the strata, there exists a natural cospecialization map $F|_s^* \to F|_t^*$ whenever the stratum containing point s lies in the closure of the stratum containing the point t.

Cospecialization construction gives us a morphism $\mathcal{A}_\mu \to \mathcal{A}_{\mu_1} \otimes \mathcal{A}_{\mu_2}$. By summing over for all μ we obtain the comultiplication morphism $\Delta \colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$.

2.12. **Schieder's deformation.** We have constructed comultiplication morphism (it was standard), now we are going to construct a multiplication morphism by constructing another deformation of $\mathcal{F}^\mu = \overline{S}_\mu \cap T_0$. Let us again start from the case $G = SL_2$, consider the following moduli space:

$$Y^n := \{\mathcal{L} \xrightarrow{\eta} \mathcal{E}_1 \xrightarrow{\varphi} \mathcal{E}_2 \xrightarrow{\zeta} \mathcal{O} \mid \deg(\mathcal{L}) = -n,\ \zeta \circ \varphi \circ \eta \text{ is an isomorphism generically}\},$$

η is an embedding of vector bundles, ζ is a surjection.

There exists a natural morphism $v\colon Y^n \to \mathbb{A}^1$ that sends (η, φ, ζ) to $\det(\varphi)$. We note now that the fiber over any point $t \neq 0$ identifies with $\overset{\circ}{Z}{}^n$, hence, we obtain a deformation of zastava. Let us describe the fiber over zero. Note that we still have a factorization morphism $Y^n \to X^{(n)}$ that sends (η, φ, ζ) to the divisor that corresponds to the composition $\zeta \circ \varphi \circ \eta$. Denote by $\mathfrak{Y}^n$ the fiber over $n \cdot 0$. It consists of the data $\mathcal{O}(-n\cdot 0) \xrightarrow{\eta} \mathcal{E}_1 \xrightarrow{\varphi} \mathcal{E}_2 \xrightarrow{\zeta} \mathcal{O}$ it maps to $\mathbb{A}^1$ via $\det(\varphi)$, let us describe the fiber over 0. We have $\det(\varphi) = 0$, hence, the morphism φ may be decomposed as $\mathcal{E}_1 \xrightarrow{\zeta'} \mathcal{O}(-n') \xrightarrow{\eta'} \mathcal{E}_2$ for some n. We note now that the data $\mathcal{O}(-n) \xrightarrow{\eta} \mathcal{E}_1 \xrightarrow{\zeta'} \mathcal{O}(-n')$ defines a point of $\mathcal{F}^{n-n'}$ and the data of $\mathcal{O}(-n') \xrightarrow{\eta'} \mathcal{E}_1 \xrightarrow{\zeta} \mathcal{O}$ defines a point of $\mathcal{F}^{n'}$, hence we obtain the isomorphism $\mathfrak{Y}^n|_0 \simeq \bigcup\limits_{n_1+n_2=n} \mathcal{F}^{n_1} \times \mathcal{F}^{n_2}$, hence, applying cospecialization construction we obtain a morphism

$$\bigoplus_{n_1+n_2=n} H_c^{\mathrm{top}}(\mathcal{F}_{n_1}, \mathbb{C}) \otimes H_c^{\mathrm{top}}(\mathcal{F}_{n_2}, \mathbb{C}) = \bigoplus_{n_1+n_2=\alpha} \mathcal{A}_{n_1} \otimes \mathcal{A}_{n_2} \to \mathcal{A}_{nl} = H_c^{\mathrm{top}}(\mathcal{F}_n, \mathbb{C}).$$

Summing over all n we obtain a multiplication morphism $\mathbf{m}\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$.

It was an SL_2-case, the general case may be treated as follows. We set $\overset{\circ}{Z} := \bigsqcup\limits_{\mu} \overset{\circ}{Z}{}^\mu$. This space classifies pairs of B-structure and U_--structure in the trivial bundle $\mathcal{E}$ that are generically transversal.

Note that it is isomorphic to the following space of maps: $\overset{\circ}{Z} = \mathrm{Maps}_{\mathrm{gen}}(X, G/(B \times U_-) \supset \mathrm{pt})$, here U_- is a unipotent radical of B_- and by $\mathrm{Maps}_{\mathrm{gen}}(X, S \supset S_0)$ we mean such maps $X \to S$ that generically go to S_0. We want to deform this space. To do so, we can deform the group G.

2.12.1. *Vinberg semigroup.* Let r be the rank of G. There exists a semigroup to be denoted Vin_G equipped with a morphism $v\colon \mathrm{Vin}_G \to \mathbb{A}^r$ such that a fiber over any point $(c_1, \ldots, c_r)$ with all c_i nonzero is isomorphic to G. This family also has a section $s\colon \mathbb{A}^r \to \mathrm{Vin}_G$.

Example 2.6. For $G = SL_2$ we have $\mathrm{Vin}_G = \mathrm{Mat}_{2\times 2}$ and the morphism v simply sends A to $\det(A)$, s sends c to $\mathrm{diag}(1, c)$.

Remark 2.13. Vinberg semigroup can be constructed as follows. Recall that the algebra of functions $\mathbb{C}[G]$ admits a natural multifiltration by the character lattice Λ_G. By the definition, Vin_G is the spectrum of the Rees algebra with respect to this filtration.

Denote by $\mathrm{Vin}_G^{\mathrm{Bruhat}}$ the $B \times U_-$-orbit of the section $s(\mathbb{A}^r)$ and finally define $Y := \mathrm{Maps}_{\mathrm{gen}}(X, \mathrm{Vin}_G/(B \times U_-) \supset \mathrm{Vin}_G^{\mathrm{Bruhat}}/(B \times U_-)$.

The scheme Y^μ can be described in the following Tannakian way:

1) two right G-torsors $\mathcal{E}_1, \mathcal{E}_2$ on X,
2) for every $\lambda^\vee \in \Lambda^{\vee +}$ a morphism $\varphi_{\lambda^\vee}\colon \mathcal{V}^{\lambda^\vee}_{\mathcal{E}_1} \to \mathcal{V}^{\lambda^\vee}_{\mathcal{E}_2}$,
3) for every $\mu^\vee \in \Lambda^\vee_+$ a number $c_{\mu^\vee}$,
4) a T-bundle $\mathcal{T}$ on X of degree $-\mu$,
5) for every $\lambda^\vee \in \Lambda^{\vee +}$, morphisms of vector bundles

$$\eta_{\lambda^\vee}\colon \mathbb{C}^{\lambda^\vee}_{\mathcal{T}} \hookrightarrow \mathcal{V}^{\lambda^\vee}_{\mathcal{E}_1},\ \zeta_{\lambda^\vee}\colon \mathcal{V}^{\lambda^\vee}_{\mathcal{E}_2} \twoheadrightarrow \mathcal{O}_X,$$

satisfying the following conditions:

a) for every $\lambda^\vee \in \Lambda^{\vee +}$, the composition

$$\zeta_{\lambda^\vee} \circ \eta_{\lambda^\vee}\colon \mathbb{C}^{\lambda^\vee}_{\mathcal{T}} \to \mathcal{O}_X$$

is an isomorphism generically.

b) The Plücker relations hold.

c) For any $\mu_1^\vee, \mu_2^\vee \in \Lambda^\vee_+$ we have $c_{\mu_1^\vee} c_{\mu_2^\vee} = c_{\mu_1^\vee + \mu_2^\vee}$.

d) For any $\lambda_1^\vee, \lambda_2^\vee, \nu^\vee \in \Lambda^{\vee +}$ such that $V^{\nu^\vee}$ enters $V^{\lambda_1^\vee} \otimes V^{\lambda_2^\vee}$ with nonzero multiplicity, we denote by $\iota\colon W^{\nu^\vee} \hookrightarrow V^{\lambda_1^\vee} \otimes V^{\lambda_2^\vee}$ the embedding of the corresponding isotypical component and by $\mathrm{pr}\colon V^{\lambda_1^\vee} \otimes V^{\lambda_2^\vee} \twoheadrightarrow W^{\nu^\vee}$ the corresponding projection. We denote by $\iota^{\mathcal{E}_1}, \mathrm{pr}^{\mathcal{E}_2}$ the corresponding morphisms between the induced vector bundles. Then we have

$$\mathrm{pr}^{\mathcal{E}_2} \circ (\varphi_{\lambda_1^\vee} \otimes \varphi_{\lambda_2^\vee}) \circ \iota^{\mathcal{E}_1} = (c_{\lambda_1^\vee + \lambda_2^\vee - \nu^\vee}) \cdot \varphi_{\nu^\vee}.$$

e) The morphism φ_0 coincides with the identity morphism, and the morphism τ_0 coincides with the identity morphism.

We have a natural morphism $Y^\mu \to \mathbb{A}^r$, we will denote by the same symbol (Y^μ) the restriction of this deformation to the diagonal $\mathbb{A}^1 \hookrightarrow \mathbb{A}^r$.

Remark 2.14. Variety Y^α is irreducible.

Remark 2.15. The morphism v is flat. To see that we note that v is surjective and Y^α is irreducible. Note also that the morphism v is not smooth in general.

2.16. **Multiplication via Schieder degeneration.** We have the factorization morphism $Y^\alpha \to X^{(\alpha)}$ we denote by $\mathfrak{Y}^\alpha$ the preimage of $\alpha \cdot 0$. Consider the restriction $v|_{\mathfrak{Y}^\alpha}\colon \mathfrak{Y}^\alpha \to \mathbb{A}^1$. The fibers $\mathfrak{Y}^\alpha|_t$ for $t \neq 0$ are isomorphic to $\mathcal{F}^\alpha$. The fiber over 0 naturally identifies with the union $\bigcup\limits_{\alpha_1+\alpha_2=\alpha} \mathcal{F}_{\alpha_1} \times \mathcal{F}_{\alpha_2}$, hence, applying cospecialization construction we obtain a morphism

$$\bigoplus_{\alpha_1+\alpha_2=\alpha} H_c^{\mathrm{top}}(\mathcal{F}_{\alpha_1}, \mathbb{C}) \otimes H_c^{\mathrm{top}}(\mathcal{F}_{\alpha_2}, \mathbb{C}) = \bigoplus_{\alpha_1+\alpha_2=\alpha} \mathcal{A}_{\alpha_1} \otimes \mathcal{A}_{\alpha_2} \to \mathcal{A}_\alpha = H_c^{\mathrm{top}}(\mathcal{F}_\alpha, \mathbb{C}).$$

Summing over all α we obtain a multiplication morphism $\mathbf{m}\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$.

3. Construction of the action

Example 3.7. Recall that the space Z^n coincides with the following set:

$$\{M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \mid \det(M) = 1, Q \text{ is monomial of degree } n, \deg(P) < n\}$$

We have the natural deformation:

$$Y^n := \{M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \mid \det(M) \in \mathbb{C}, Q \text{ is monomial of degree } n, \deg(P) < n\}$$

equipped with a morphism $Y^n \to \mathbb{A}^1$, $M \mapsto \det(M)$. This is exactly Shieder's degeneration!

We now fix a perverse sheaf $\mathcal{P} \in \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ and recall a bialgebra $\mathcal{A}$ of the last Section. We set $V := H^*(\mathrm{Gr}_G, \mathcal{P})$. Our goal is to construct an action $\mathcal{A} \curvearrowright V$. To do so, we first recall that $V = \bigoplus_{\mu \in \Lambda_G} V_\mu$ – $T^\vee$-weight decomposition and that $V_\mu = H^{\langle 2\rho^\vee, \mu\rangle}(S_\mu, \mathcal{P})$, here, $2\rho^\vee$ is the sum of positive roots. So, we must construct maps $\mathcal{A}_{\alpha_1} \otimes V_{\alpha_2} \to V_\alpha$, $\alpha = \alpha_1 + \alpha_2$. We will do it by deforming ind-schemes S_α (we start from deforming Gr_G the whole Grassmannian).

3.1. **Main construction.** We start from the simplest case $G = SL_2$. The general case will be treated completely analogously via Tannakian approach. Recall that Gr_G is the moduli space of pairs $(\mathcal{E}, \sigma)$, where $\mathcal{E}$ is a G-bundle on X and σ is a trivialization of $\mathcal{E}$ away from $\{x\}$. We note that this moduli space coincides with the moduli space of the following data: $\{\mathcal{O} \xrightarrow{\eta} \mathcal{E} \xrightarrow{\zeta} \mathcal{O}\}$, where η, ζ are rational morphisms with poles at zero such that $(\zeta \circ \eta)|_{X \setminus \{x\}} = Id$. Let us comment that a rational morphism $\mathcal{O} \to \mathcal{E}$ with poles at zero is by the definition an element of the inductive limit $\varinjlim\{\mathcal{O}(-n \cdot 0) \to \mathcal{E}\}$ (for ζ the definition is analogous).

We are now ready to define our main player – the deformation of Gr_G:

$$\widetilde{\mathrm{Gr}}_G := \{\mathcal{O} \xrightarrow{\eta} \mathcal{E}_1 \xrightarrow{\varphi} \mathcal{E}_2 \xrightarrow{\zeta} \mathcal{O} \mid \eta, \zeta - \text{rational with pole at zero}, \ (\zeta \circ \varphi \circ \eta)|_{\mathbb{P}^1 \setminus \{0\}} = Id\}.$$

Let us define now the degeneration $\widetilde{S}_n \to \mathbb{A}^1$. To do so, we recall the modular approach to S_n:

$$S_n = \{\mathcal{O}(-n \cdot 0) \xrightarrow{\eta} \mathcal{E} \xrightarrow{\zeta} \mathcal{O} \mid \zeta - \text{rational}, (\zeta \circ \eta)|_{(\mathbb{P}^1 \setminus \{0\})} = Id\}.$$

We now define degeneration

$$\widetilde{S}_n = \{\mathcal{O}(-n \cdot 0) \xrightarrow{\eta} \mathcal{E}_1 \xrightarrow{\varphi} \mathcal{E}_2 \xrightarrow{\zeta} \mathcal{O} \mid \zeta - \text{rational}, (\zeta \circ \varphi \circ \eta)|_{(\mathbb{P}^1 \setminus \{0\})} = Id\}.$$

Let us give the Tannakian definition of the ind-scheme $\overline{S}_\nu \subset \mathrm{Gr}_G$. The corresponding functor of points associates to a scheme S

1) a G-bundle $\mathcal{E}$ on X,

2) for every $\lambda^\vee \in \Lambda^{\vee +}$, morphisms of sheaves $\eta_{\lambda^\vee} : \mathcal{O}_X(-\langle \lambda^\vee, \nu\rangle \cdot x) \to \mathcal{V}_{\mathcal{E}}^{\lambda^\vee}$ and rational morphisms $\zeta_{\lambda^\vee} : \mathcal{V}_{\mathcal{E}}^{\lambda^\vee} \to \mathcal{O}_X$ regular on $(X \setminus \{x\})$, satisfying Plücker relations.

The deformations $\widetilde{\mathrm{Gr}}_G$, $\widetilde{S}_\mu$ can be now defined analogously to Y^μ above.

Main property of the deformation $\widetilde{\mathrm{Gr}}_G$: for any $t \neq 0$, the fiber $(\widetilde{\mathrm{Gr}}_G)|_t$ is isomorphic to Gr_G, for $t = 0$ we have $(\widetilde{\mathrm{Gr}}_G)|_0 = \bigcup_{\mu_1 + \mu_2 = \mu} \overline{T}_{\mu_1} \times \overline{S}_{\mu_2}$.

Analogously, for any $t \neq 0$, the fiber $(\widetilde{S}_\mu)|_t$ is isomorphic to S_μ, for $t = 0$ we have $(\widetilde{S}_\mu)|_0 = \bigcup_{\mu_1 + \mu_2 = \mu} \mathcal{F}^{\mu_1} \times S_{\mu_2}$.

3.2. **Action.** To define the action via our deformation we must construct a sheaf $\widetilde{\mathcal{P}}$ on $\widetilde{S}_\mu$ that restricts to $\mathbb{C} \boxtimes (\mathcal{P}|_{S_{\mu_2}})$ on zero fiber and to $\mathcal{P}|_{S_\mu}$ on the fiber over 1.

To do so we start from the constructing of the closed embedding $\iota\colon \widetilde{S}_\mu \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G$. It simply sends

$$\mathcal{O} \xrightarrow{\eta} \mathcal{E}_1 \xrightarrow{\varphi} \mathcal{E}_2 \xrightarrow{\zeta} \mathcal{O}$$

to the pair

$$(\mathcal{O} \xrightarrow{\varphi\circ\eta} \mathcal{E}_1 \xrightarrow{\zeta} \mathcal{O}\,,\, \mathcal{O} \xrightarrow{\eta} \mathcal{E}_1 \xrightarrow{\varphi\circ\zeta} \mathcal{O}) \in \mathrm{Gr}_G \times \mathrm{Gr}_G\,.$$

Lemma 3.8. *The morphism ι is a closed embedding.*

Define $\tilde{\mathcal{P}} := \iota^*(\mathbb{C} \boxtimes \mathcal{P}|_{S_\mu})$.

Remark 3.3. We note that

We have constructed a sheaf $\widetilde{\mathcal{P}}$ to $\tilde{S}_\mu$. It is easy to see that the restriction $\widetilde{\mathcal{P}}|_{(\widetilde{S}_\mu)_t}$ is isomorphic to $\mathcal{P}|_{S_\mu}$ and the restriction of $\widetilde{\mathcal{P}}$ to $\mathcal{F}^{\mu_1} \times S_{\mu_2}$ is isomorphic to $\mathbb{C} \boxtimes (\mathcal{P}|_{S_{\mu_2}})$.

The cospecialization construction defines a morphism

$$\bigoplus_{\mu_1+\mu_2=\mu} \mathcal{A}_{\mu_1} \otimes V_{\mu_2} \to V_\mu.$$

After summing over all μ we obtain a morphism $\mathrm{act}\colon \mathcal{A} \otimes V \to V$.

Let us summarize our results:

Theorem 3.9. *a) The bialgebra $(\mathcal{A}, \Delta, \mathbf{m})$ is isomorphic to the universal enveloping algebra $U(\mathfrak{n}^\vee)$.*

b) The natural action $U(\mathfrak{n}^\vee) \curvearrowright V$ geometrically is given by cospecialization construction above and coincides with the morphism act *after the identification $\mathcal{A} \simeq U(\mathfrak{n}^\vee)$.*

3.4. **Sketch of proof.** The proof goes as follows.

3.4.1. *First step.* First of all we must check that the morphism $\mathrm{act}\colon \mathcal{A} \otimes V \to V$ actually defines an action $\mathcal{A} \curvearrowright V$. To do so, we must show that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes V & \xrightarrow{\mathbf{m}\otimes\mathrm{Id}} & \mathcal{A} \otimes V \\ \downarrow{\scriptstyle \mathrm{Id}\otimes\mathrm{act}} & & \downarrow{\scriptstyle \mathrm{act}} \\ \mathcal{A} \otimes V & \xrightarrow{\mathrm{act}} & V. \end{array}$$

To prove this, we construct a family over $\mathbb{A}^2$ and a certain complex on this family such that the composition $\mathrm{act} \circ (\mathbf{m} \otimes \mathrm{Id})$ coincides with the cospecialization morphism for this complex from $(0,0)$ to $(1,0)$ and then to $(1,1)$. On the other hand the composition $\mathrm{act} \circ (\mathrm{Id} \otimes \mathrm{act})$ coincides with the cospecialization morphism for our complex from $(0,0)$ to $(0,1)$ and then to $(1,1)$.

Recall now that given a stratified variety S and a complex F on S which is constant on the strata, there exists a natural cospecialization map $F|_s^* \to F|_t^*$ whenever the stratum containing point s lies in the closure of the stratum containing the point t.

We note now that both the compositions of cospecializations above tautologically coincide with the cospecialization morphism from $(0,0)$ to $(1,1)$ in the whole family over $\mathbb{A}^2$.

3.4.2. *Second step.* Our aim now is to prove that the action that we have constructed comes from the natural action $U(\mathfrak{g}^\vee) \curvearrowright V$ i.e. that there exists a morphism of algebras $\mathcal{A} \to U(\mathfrak{g}^\vee)$ that commutes with our actions. From the general Tannakian formalism, to do so it is enough to prove that our action is compatible with the tensor structure on $\mathrm{Rep}(G^\vee)$ i.e. that for any two finite-dimensional $G^\vee$-modules V, W the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A} \otimes V \otimes W & \xrightarrow{\mathrm{act}_{V\otimes W}} & V \otimes W \\ \downarrow{\scriptstyle \Delta\otimes\mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathcal{A} \otimes \mathcal{A} \otimes V \otimes W & \xrightarrow{(\mathrm{act}_V \otimes \mathrm{act}_W)\circ(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})} & V \otimes W \end{array} \tag{3.1}$$

where the morphism $\tau\colon \mathcal{A} \otimes V \to V \otimes \mathcal{A}$ sends $a \otimes b$ to $b \otimes a$. For proving it we again construct a two-parametric family and a complex such that both compositions in our diagram coincide with the cospecialization morphism from $(0,0)$ to $(1,1)$. Let us briefly describe the construction. Again let us do it for SL_2, in general case we just have to use a Tannakian approach. Recall that the family $\widetilde{S}_n$ that gave us the action $\mathcal{A} \curvearrowright V$ was the moduli space

$$\widetilde{S}_n = \{\mathcal{O}(-n\cdot 0) \xrightarrow{\eta} \mathcal{E}_1 \xrightarrow{\varphi} \mathcal{E}_2 \xrightarrow{\zeta} \mathcal{O} \,|\, \zeta - \text{can have poles at zero}, (\zeta\circ\varphi\circ\eta)|_{\mathbb{P}^1\setminus\{0\}} = \mathrm{Id}\}.$$

We now deform it in the following way:

$$\begin{aligned}\widetilde{\widetilde{S}}_n = \{\mathcal{O}(-n\cdot 0) \xrightarrow{\eta} \mathcal{E}_1 \xrightarrow{\varphi_1} \mathcal{E}_2 \xrightarrow{\varphi_2} \mathcal{E}_3 \xrightarrow{\zeta} \mathcal{O} \,|\\ |\, \zeta - \text{can have poles at zero}, (\zeta\circ\varphi_1\circ\varphi_2\circ\eta)|_{\mathbb{P}^1\setminus\{0\}} = \mathrm{Id}\}.\end{aligned}$$

We obtain the desired two-parametric family, the morphism $\widetilde{\pi}\colon \widetilde{\widetilde{S}}_n \to \mathbb{A}^2$ is given by $(\eta, \varphi_1, \varphi_2, \zeta) \mapsto (\det\varphi_1, \det\varphi_2)$. We have a natural closed embedding $\widetilde{\iota}\colon \widetilde{\widetilde{S}}_n \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathrm{Gr}_G$ and we consider a complex $\tilde{\tilde{\mathcal{P}}} := \widetilde{\iota}^*(\mathbb{C} \boxtimes \mathbb{C} \boxtimes \mathcal{P})$.

After obtaining a morphism $U(\mathfrak{a}) \to U(\mathfrak{g}^\vee)$ it is easy to see that it is injective and graded dimensions coincide, hence, we obtain the isomorphism.

Remark 3.5. According to the Tannakian formalism, group $G^\vee$ coincides with the group of (tensor) authomorphisms of the fiber functor. Analogously, the Lie algebra $\mathfrak{g}^\vee$ coincides with the set of endomorphisms E of the fiber functor F such that for any V, W the following diagram is commutative:

$$\begin{array}{ccc} F(V) \otimes F(W) & \xrightarrow{E(V)\otimes\mathrm{Id} + \mathrm{Id}\otimes E(W)} & F(V) \otimes F(W) \\ \downarrow & & \downarrow \\ F(V \otimes W) & \xrightarrow{E(V\otimes W)} & F(V \otimes W). \end{array}$$

We now consider the Lie algebra $\mathfrak{a} \subset \mathcal{A}$, consisting of primitive elements. It follows from 3.1 and the observation above that we have a Lie algebra homomorphism $\mathfrak{a} \to \mathfrak{g}^\vee$ that induces a morphism $U(\mathfrak{a}) \to U(\mathfrak{g}^\vee)$.

4. Possible generalizations and Drinfeld-Gaitssgory deformations

4.1. **Drinfeld-Gaitsgory interpolation.** Let us start this Section from one general construction due to Drinfeld-Gaitsgory. Let Z be an algebraic variety equipped with a $\mathbb{C}^\times$-action. Let us also denote by $\mathbb{X}$ the variety $\mathbb{A}^2$ with the following action of $\mathbb{C}^\times$: $(x, y) \mapsto (tx, t^{-1}y$. Note that the natural morphism $\mathbb{X} \to \mathbb{A}^1$ that sends (x, y) to xy is equivariant with respect to the trivial $\mathbb{C}^\times$-action on $\mathbb{A}^1$.

Let us define a scheme $\widetilde{Z}^{\mathrm{dg}}$ (it will be certain scheme over $\mathbb{A}^1$). Fix an $\mathbb{A}^1$-scheme S. Define $\mathrm{Maps}_{\mathbb{A}^1}(S, \widetilde{Z}^{\mathrm{dg}}) := \mathrm{Maps}(S \underset{\mathbb{A}^1}{\times} \mathbb{X}, Z)^{\mathbb{G}_m}$.

4.1.1. *Main properties.* By the definition we have a morphism $\widetilde{Z}^{\mathrm{dg}} \to \mathbb{A}^1$. We denote by $\widetilde{Z}^{\mathrm{dg}}_t$ the fiber over zero.

Proposition 4.10. *1) We have $\widetilde{Z}^{\mathrm{dg}}_t \simeq Z$ for $t \neq 0$.*

2) We have $\widetilde{Z}^{\mathrm{dg}}_0 \simeq Z^+ \underset{Z^{\mathbb{C}^\times}}{\times} Z^-$, where $Z^+ := \{x \in Z \,|\, \exists \lim\limits_{t\to 0} t \cdot x\}$

4.2. **Drinfeld-Gaitsgory interpolation of Gr_G.** Recall the reductive group G, maximal torus $T \subset G$ and the affine Grassmannian Gr_G, consider the $\mathbb{C}^\times$-action via $2\rho\colon \mathbb{C}^\times \to T$, here T acts by left multiplication. We obtain the interpolation $\widetilde{\mathrm{Gr}}^{\mathrm{dg}}_G$.

Relation between $\widetilde{\mathrm{Gr}}^{\mathrm{dg}}_G$ and $\widetilde{\mathrm{Gr}}_G$ is the following. There exists an open embedding $\widetilde{\mathrm{Gr}}^{\mathrm{dg}}_G \hookrightarrow \widetilde{\mathrm{Gr}}_G$ that is isomorphism away from zero fiber, on the level of zero fiber it coincides with the natural embedding $\bigsqcup\limits_{\mu} T_\mu \times S_\mu \hookrightarrow \bigcup\limits_{\mu} \overline{T}_\mu \times \overline{S}_\mu$.

5. Questions

5.1. **Quantum universal enveloping algebra.** Is it possible to geometrically construct $U_q(\mathfrak{n}^\vee)$ in the same manner?

5.2. **Drinfeld-Gaitsgory interpolations.** General Drinfeld-Gaitsgory interpolation construction?

5.3. **Kac-Moody Lie algebras.** Generalization to affine (Kac-Moody) Lie algebras?